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On the instability of rapidly rotating shear flows to non-axisymmetric disturbances

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The stability is considered of the flow with velocity components

$$\{0, \Omega r[1+O(\epsilon^2)],\ 2\epsilon\Omega r_0 f(r/r_0)\}$$

(where $f(x)$ is a function of order one) in cylindrical polar co-ordinates (r, ϕ, z), bounded by the rigid cylinders $r/r_0 = x_1$ and $r/r_0 = 1$ $(0 \leqslant x_1 < 1)$. When $\epsilon \ll 1$, the flow is shown to be unstable to non-axisymmetric inviscid disturbances of sufficiently large axial wavelength. The case of Poiseuille flow in a rotating pipe is considered in more detail, and the growth rate of the most rapidly growing disturbance is found to be $2\epsilon\Omega$.

1. Description of problem

It is well known that pure solid body rotation in an incompressible fluid is stable to all infinitesimal disturbances. It is sometimes inferred from this that flows which deviate only slightly from solid body rotation, such as a slow axial flow in a rapidly rotating pipe, must be equally stable. However, Ludwieg (1961) showed that the flow given in cylindrical polar co-ordinates (r, ϕ, z) by the velocity components $\{0, V_0[1+C_\phi(x-1)], W_0[1+C_z(x-1)]\}$, where $x = r/r_0$, where r_0, V_0, W_0, C_ϕ, C_z are constants, and where the boundaries of the flow are the rigid walls of the narrow cylindrical annulus $1-\delta \leqslant x \leqslant 1+\delta$ $(\delta \ll 1)$, is unstable to infinitesimal inviscid disturbances if

$$(1-C_\phi)(1-C_\phi^2) - (\tfrac{5}{3}-C_\phi)\,C_z^2 < 0. \tag{1}$$

In particular, if $C_\phi = 1$ (solid body rotation), the flow is unstable for all non-zero C_z, however small. Thus, subject to the narrow-gap approximation, an arbitrarily small axial shear is sufficient to destabilize solid body rotation. In this note we show that the same conclusion may be reached without the narrow gap approximation.

Consider the basic flow $[0, V(r), W(r)]$ bounded by the rigid cylindrical annulus $r_1 \leqslant r \leqslant r_0$, where $r_0 > r_1 \geqslant 0$. Assuming infinitesimal perturbations to this basic flow, we can linearize the equations of motion, and eliminate all perturbation quantities except the radial velocity component, which we suppose to be written

$$u(r)\exp\{i(\sigma t + n\phi + kz)\},$$

where k is positive and n is an integer. In an inviscid fluid, the resulting single equation for $u(r)$ is (Howard & Gupta 1962, equation (18)):

$$\gamma^2 \frac{d}{dr}\left[\frac{r}{n^2+k^2r^2}\frac{d}{dr}(ru)\right] - u\left\{\gamma^2 + \gamma r\frac{d}{dr}\left[\frac{r}{n^2+k^2r^2}\left(\frac{d\gamma}{dr}+\frac{2nV}{r^2}\right)\right]\right. \\ \left. -\frac{2kV}{n^2+k^2r^2}\left[k\frac{d}{dr}(rV) - n\frac{dW}{dr}\right]\right\} = 0, \qquad (2)$$

where
$$\gamma = \sigma + \frac{nV(r)}{r} + kW(r). \qquad (3)$$

The boundary conditions are

$$u(r_1) = u(r_0) = 0. \qquad (4)$$

The problem is thus seen as a search for eigenvalues of the quantity σ; instability will result if the imaginary part of σ (and hence γ) is negative for any pair of wave-numbers (n, k). From equations (2)–(4), Howard & Gupta (1962) showed that a necessary condition for instability is

$$\frac{k^2}{r^3}\frac{d}{dr}(r^2V^2) - \frac{2knV}{r^2}\frac{dW}{dr} - \frac{1}{4}\left[\frac{d\gamma}{dr}\right]^2 < 0 \qquad (5)$$

for some r in the range (r_1, r_0). Hence when $V = \Omega r$ (solid body rotation), the flow is stable *to axisymmetric disturbances* ($n = 0$) if $|dW/dr| < 4\Omega$ everywhere.

2. Proof of instability

Let us now non-dimensionalize the problem by the following transformations:

$$x = \frac{r}{r_0}, \quad V(r) = \Omega r_0 x g(x), \quad W(r) = W_0 f(x), \quad y = \frac{ru}{r_0 W_0},$$
$$\alpha = kr_0 > 0, \quad \epsilon = \frac{W_0}{2\Omega r_0}, \quad \omega = \frac{\gamma}{2\Omega} = \frac{\sigma}{2\Omega} + \frac{n}{2}g(x) + \epsilon\alpha f(x), \qquad (6)$$

where the functions $f(x)$ and $g(x)$ are of order one, and without loss of generality we take $g(1) = 1, f(1) = 0$. For solid body rotation $g(x) \equiv 1$. Equation (2) now becomes

$$\frac{d}{dx}\left(\frac{xy'}{n^2+\alpha^2x^2}\right) + y\left\{-\frac{1}{x} - \frac{1}{\omega}\frac{d}{dx}\left(\frac{ng+x\omega'}{n^2+\alpha^2x^2}\right)\right. \\ \left. + \frac{\alpha g}{\omega^2(n^2+\alpha^2x^2)}\left[\frac{\alpha}{2}\frac{d}{dx}(x^2g) - \epsilon n f'\right]\right\} = 0, \qquad (7)$$

where a prime denotes differentiation with respect to x. The boundary conditions are $y(x_1) = y(1) = 0$, where $x_1 = r_1/r_0$ (if $x_1 = 0$, the left-hand condition is $y(x) = o(x)$ as $x \to 0$). Howard & Gupta's condition (5) is, in non-dimensional terms,

$$g^2 + \tfrac{1}{2}xgg' - \frac{\epsilon ngf'}{\alpha x} - \left(\frac{\omega'}{2\alpha}\right)^2 < 0. \qquad (8)$$

Since n may be chosen to take either sign, the quantity nf' may always be made positive at any x where $f'(x) \neq 0$. Thus condition (8) can be satisfied at any such

x and for arbitrarily small values of ϵ, as long as $|\alpha/n|$ is sufficiently small. We are primarily interested in the destabilising influence of a small axial shear superimposed on the rotation, so we shall indeed consider only small values of ϵ, and may therefore restrict our attention to small values of $|\alpha/n|$, setting $\alpha = \beta\epsilon n$, where $\beta = O(1)$, and $\beta n > 0$ (since $\alpha > 0$).

It is possible to make equation (6) entirely tractable if we impose one further restriction, this time on the basic swirl: assume that $g(x)$ is given by

$$g(x) \equiv 1+\epsilon^2 h(x), \tag{9}$$

where h is of order one, and $h(1) = 0$. The basic flow defined by (6) and (9) is thus solid body rotation, perturbed by an axial shear flow of order ϵ ($\ll 1$) and an azimuthal correction of order ϵ^2, and is evidently stable to axisymmetric disturbances since

$$|dW/dr| = 2\epsilon\Omega|f'| \ll 4\Omega.$$

The quantity ω is now a constant to within order ϵ^2

$$\omega = \frac{\sigma}{2\Omega}+\frac{n}{2}+n\epsilon^2\left(\frac{h}{2}+\beta f\right) = \omega_0+O(\epsilon^2) \tag{10}$$

say, so that, if $O(\epsilon^2)$ is neglected with respect to unity, equation (7) reduces to

$$\frac{1}{n^2}\frac{d}{dx}(xy')+y\left\{-\frac{1}{x}-\frac{\epsilon^2}{n\omega_0}(\tfrac{3}{2}h'+\tfrac{1}{2}xh''+\beta f'+\beta xf''-2\beta^2 x)+\frac{\epsilon^2\beta x}{\omega_0^2}\left(\beta-\frac{f'}{x}\right)\right\} = 0 \tag{11}$$

subject to the boundary conditions $y(x_1) = y(1) = 0$. If we are permitted to neglect the second term in the curly bracket, we are left with a Sturm–Liouville problem. In these circumstances the characteristic values for ω_0^2 are all real, and some of them are negative (implying instability) if

$$\beta(\beta-f'/x) < 0 \tag{12}$$

for some x in $(x_1, 1)$. When (12) is satisfied, the negative eigenvalues of ω_0^2 in general have order ϵ^2, although the largest of them (corresponding to the most rapidly growing disturbance) may have a higher order. In general, therefore, there exist unstable modes for which $|\omega_0^2| = O(\epsilon^2)$ and in that case the second term in the curly bracket of (11) is of order ϵ, and its neglect is justified. Thus a necessary and sufficient condition for the flow to be unstable to a disturbance characterized by the non-dimensional wave-numbers (α, n), where $\alpha = \beta\epsilon n$, is given by (12), and we can see that a β can be chosen to satisfy (12) at all x where $f'(x) \neq 0$. The flow is therefore unstable.

If the radial scale of variations in the basic axial velocity is much smaller than the dimensions of the container, we should use that scale for r_0 in (6), and let the outer boundary be $r/r_0 = x_2 \gg 1$. The theory still goes through as long as $\alpha x_2 = \beta\epsilon n x_2 \ll 1$.

Finally we notice that there is one flow for which the problem reduces to Sturm–Liouville form without the assumption of small ϵ. This is the flow for which

$$f(x) \equiv \frac{2\epsilon\alpha}{n}[1-g(x)]$$

to make ω constant, and

$$g(x) \equiv \frac{n^2+\alpha^2 x^2}{n^2+\alpha^2}$$

to make the $1/\omega$ term in (7) disappear. In this case, however, the coefficient of y/ω^2 is positive, and the flow is stable.

3. Example: Poiseuille flow in a rotating pipe

A particular example of the flows discussed above is Poiseuille flow in a rotating pipe, which is relatively simple to produce experimentally. Here

$$f(x) \equiv 1-x^2, g(x) \equiv 1, x_1 = 0.$$

The condition (12) demonstrates that the flow is unstable to all disturbances for which $-2 < \beta < 0$, i.e. for which $n < -1$ ($n = -1$ is prohibited by the boundary condition $y(x) = o(x)$ as $x \to 0$) and $0 < \alpha < -2\epsilon n$. We may calculate the growth rate of unstable disturbances by actually solving equation (11), which here reduces to

$$\frac{1}{n^2}\frac{d}{dx}(xy')+y\left\{-\frac{1}{x}+\frac{x\epsilon^2\beta(\beta+2)}{\omega_0^2}\right\} = 0. \tag{13}$$

The solution of (13) satisfying the condition $y(x) = o(x)$ as $x \to 0$ is

$$y = J_m(\lambda x),$$

where $m = -n > 1$, $\lambda^2 = \epsilon^2 m^2 \beta(\beta+2)/\omega_0^2$ and J_m is the Bessel function of the first kind with order m. The boundary condition $y(1) = 0$ shows that the eigenvalues for λ (and hence ω_0) are given by the equation

$$J_m(\lambda) = 0. \tag{14}$$

For given values of m and β in the unstable range, the largest value of $(-\omega_0^2)$ (i.e. the highest growth rate) is given by the smallest value of λ satisfying (14), that is, by the first zero $j_{m,1}$ of the Bessel function. For all values of m this first zero is greater than m, and for large values of m it is given asymptotically by

$$j_{m,1} \sim m + 1{\cdot}86 m^{\frac{1}{3}} + O(m^{-\frac{1}{3}}) \tag{15}$$

(Watson 1944, pp. 516 *et seq.*). Thus the corresponding value of ω_0^2 is given by

$$\omega_0^2 = \frac{\epsilon^2 m^2 \beta(\beta+2)}{j_{m,1}^2} \sim \epsilon^2\beta(\beta+2)[1+O(m^{-\frac{2}{3}})]. \tag{16}$$

Since the second term in (15) is positive, that in (16) is negative, and greater values of $(-\omega_0^2)$, for given β, occur for larger m. Also the maximum value of $-\beta(\beta+2)$ occurs for $\beta = -1$, so the most rapidly growing disturbance is given by $\beta = -1$ and $m \to \infty$, and then $\omega_0^2 = -\epsilon^2$. In dimensional terms this means that

$$\sigma = -n\Omega \pm i2\epsilon\Omega + O(\epsilon^2) \tag{17}$$

from (10), and the growth rate of the most rapidly growing disturbance is $2\epsilon\Omega$.

A non-zero but small viscosity ν will undoubtedly have a considerable effect on the above theory. However, if the Reynolds number $W_0 r_0/\nu$ is sufficiently large, it is unlikely to alter the prediction of instability, by the above mechanism, for some non-zero values of n. Its particular effect will be to stabilize those disturbances with a high wave-number ($|n| \gg 1$), so that the most rapidly growing disturbance will occur at a finite value of $|n|$, with a growth rate less than that predicted in (17).

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